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Translational free random walk of spins in the presence of a parabolic magnetic field

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Abstract

The free random walk approach has been used to analyze the attenuation of the NMR signal due to spin dephasing in the presence of a constant and pulsed parabolic magnetic field. The spin echo sequence was chosen to examine the attenuation of the NMR signal resulting from self-diffusion. In the framework of the gaussian approach, the long-time limit calculations predict more pronounced diffusion weighting for the parabolic field than for linear magnetic field. Analytical results were obtained and compared with those from other approaches based on a variety of different of approximations.

Keywords: Diffusion attenuation; Free random walk; Markov chains; Parabolic magnetic field; Spin echo

1. Introduction

The use of linearly varying magnetic fields has been a critical component of diffusion NMR experiments, especially in the light of the growing applications of NMR in medicine. The consequences of magnetic field imperfections can be divided into geometrical distortions and inaccurate quantifications of physical properties. The first is a consequence of B_0 inhomogeneity, eddy currents, and differences in susceptibility. The second is often a result of different assumptions utilized in system modeling. Simulations and experiments show that non-uniformity induces over- and underestimation (10–30%) in diffusion coefficients [1] and cannot be neglected in anisotropic diffusion tensor calculations. Non-uniformities of magnetic field gradients can cause serious artifacts in diffusion imaging and lead to image warping and spatially dependent errors in the direction and magnitude of diffusion encoding. As a consequence, fiber tracking analyses [2] become inaccurate and unreliable. Efficient correction algorithms are in the process of being developed and have not yet been implemented on clinical NMR scanners. Until recently, however, the influence of magnetic field imperfections in diffusion experiments has not been fully appreciated. Moreover, vigorous interest has grown in the theoretical and practical consideration of spatially non-linear magnetic fields, in particular the second order (parabolic) magnetic field as a means of increasing the diffusion weighting effect.

One of the most important applications of diffusion imaging in the medical arena is diffusion tensor imaging and fiber tracking in the brain. This application has advanced significantly in recent years through advances in gradient coil design and performance. Improved performance, that is maximum gradient strength and maximum slew rate, has been achieved through various compromises. One of these compromises is a reduction in the volume over which the gradients are linear. It has been shown recently [1] that by expressing the

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gradient field using a spherical harmonic expansion, the contribution of the gradient non-uniformities may be characterized. It has been shown [1] that the errors in the absolute diffusivity, for example, ranged from about −10% to +20%. This is clear evidence that the influence of higher-order gradient magnetic fields can be of great importance and account must be taken of these factors.

In the current work, we present an accurate solution to the problem of using the free random walk approach [3] in a parabolic magnetic field and prove that a more pronounced signal decay envelope occurs due to diffusion. Our theoretical results agree well with those of other approximation methods, such as the variation of local field method [4–6], the gaussian phase approximation [7], the Green's function method [8], and the propagator method [9,10].

The model system we consider consists of a group of nuclear spins confined in a magnetic field. The field is composed of a homogeneous Zeeman field, B_0, and the parabolic magnetic field, B_2, which can be either static or pulsed. The diffusion process is modeled by allowing the spins to execute a one-dimensional Markov chain random walk [11] in an infinite, homogeneous isotropic medium. Following the method proposed by Carr and Purcell [3], the self-diffusion of the particle is pictured as a succession of infinitesimal and discrete hops with motion resolved along the direction of the centric parabolic magnetic field (Fig. 1A). Motion in other dimensions is treated as independent and does not influence the coherence of the transverse magnetization. Pythagoras's theorem can be applied to calculate the total displacement. We describe the spin displacements by discrete random variables with each spin accumulating phase in an amount determined by the path it takes through the inhomogeneous field. The phase of a spin at any time can be expressed in terms of its initial position and the random variables that characterize its movement. We make the representation of the unknown distribution function of random shifts in phase as a series over a small parameter on the assumption that that the second moment dominates [12]. The results obtained are applicable in diffusion imaging and also in a series of problems associated with the calculation of signal decay by diffusion in porous media (in the case of unrestricted "short-time" diffusion).

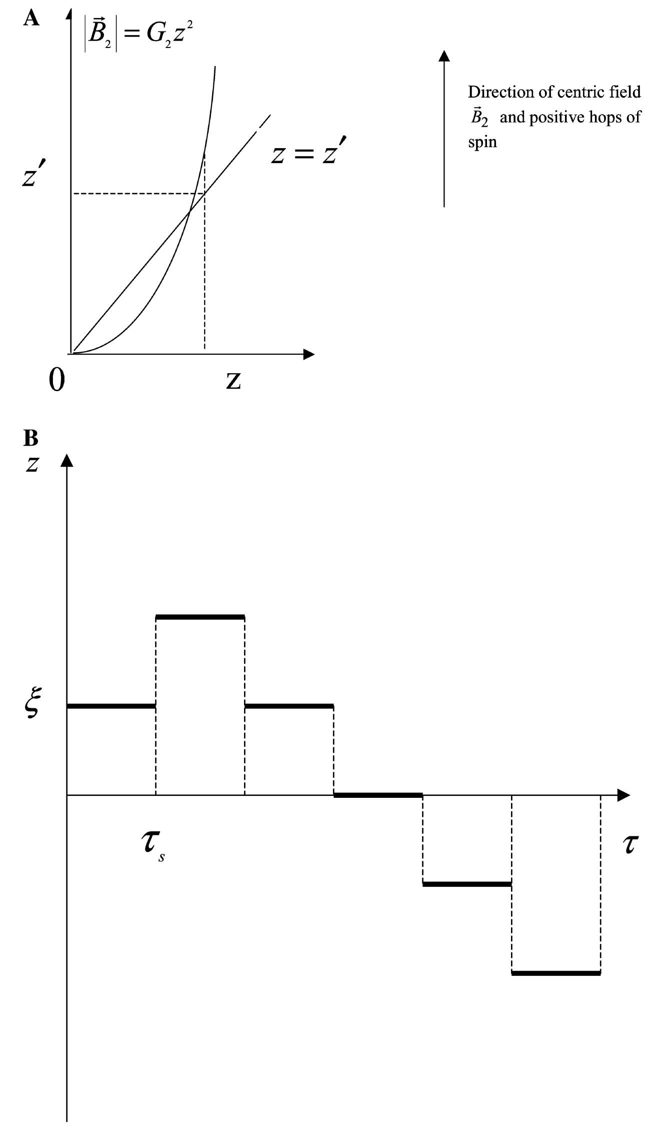

Fig. 1. (A) Schematic representation of the absolute value and direction of a centric parabolic magnetic field $\vec{B}_2$. G_2 characterizes the intensity of parabolic magnetic field. With regard to the particular question of interest here, the directions $\vec{B}_0$, $\vec{B}_2$ and the direction of the positive hops are aligned. (B) Brownian motion of a particle with step $\xi \ll R$ along z axis during a mean time, τ_s between steps.

2. Model and assumptions

Let the nuclear spins be confined to a region which occupies the interval $-R < z < R$ along the z axis. The end surfaces of the sample are perpendicular to the axis and its shape and area play no role in the following derivations. The normalized spin density along z is denoted by $\rho(z_0) = \delta(z_0)$, which is valid when considering self-diffusion. The magnitude of the Zeeman field, B_0, is much greater than the magnitude of the non-linear field $B_2(z) = G_2 z^2$, where G_2 is a coefficient of the parabolic magnetic field. For the sake of simplicity we have taken the two fields, B_0 and B_2, to be parallel to each other; this is illustrated in Fig. 1A.

Let the diffusion process be modeled by the one-dimensional Markov random walk with a step $\xi \ll R$ along the z-axis and a mean time, τ_s, between steps (Fig. 1B). The movement of the spins obeys the following rule: each spin remains at some position for a fixed period τ_s, then instantly jumps to a new position whose z coordinate differs from the previous one by a fixed amount ξ. The hops occur in the positive or the negative direction with equal probability. This is described by setting $z[(j+1)\tau_s] = z[(j\tau_s)] + \xi a_{j+1}$, where j is the number of jumps, $a_j = \{+1, -1\}$, $j = \overline{1,n}$ and $a_0 = 0$, and a_j is a random variable with a probability distribution, P, given by $P(\{a_j = \pm 1\}) = 1/2$. Suppose the starting position of the spin is $z_0 = l\xi$. Then, after time $\tau = n\tau_s$ and n jumps, the spin arrives at $z(n\tau_s) = z_0 + \xi\sum_{i=1}^{n} a_i$. The spin at this time will be in a magnetic field of $B_0 + G_2(z_0 + \xi\sum_{i=1}^{n} a_i)^2$ with the local Larmor frequency $\omega(n\tau_s) = \gamma(B_0 + G_2 z_0^2) + \Delta\omega$, where

$$\Delta\omega(n\tau_s) = \gamma G_2 \left[2z_0\xi \sum_{k=0}^{n} a_k + \left(\xi \sum_{i=0}^{n} a_i \right)^2 \right]. \tag{1}$$

This means that the local frequency, $\omega(n\tau_s)$, consists of a spatially dependent contribution, $\gamma(B_0 + G_2 z_0^2)$, and a randomly fluctuating contribution, $\Delta\omega$ (Fig. 2A). The fluctuating frequency change can be simplified as $\Delta\omega(n\tau_s) = \gamma G_2 \xi^2 \sum_{i=1}^{n} A_i$ with $A_i = 2a_i(\sum_{k=0}^{i-1} a_k) + a_i^2 + 2l(\sum_{k'=0}^{i} a_{k'})$ (Fig. 2B). Therefore, the cumulative phase change after time τ is

$$\Delta\varphi(n\tau_s) = \gamma G_2 \tau_s \xi^2 \sum_{m=1}^{n} c_m A_m, \tag{2}$$

where $c_m = n + 1 - m$, $m = \overline{1,n}$. An explanation of Eq. (2) can be found in [14]. A summation over index m is illustrated in Fig. 2A. The sum of jump units a_j along the row squared is proportional to the precessional frequency change Eq. (1). The sign at the end of each row indicates whether the jump after time τ is positive or negative with respect to the parabolic magnetic field direction. The cumulative phase shift, $\Delta\varphi$, at time τ in the parabolic magnetic field is given by the sum within the shaded triangle (Fig. 2A). The same summation can be rearranged (Eq. (2)) and represented by the sum of factors $A_i(\cdot)$ (Fig. 2B). The advantage of this treatment is that the view of the sum of Eq. (2) maximally corresponds to the case of a linear magnetic field and all rules of summation, developed for the linear magnetic field, can be applied automatically to the parabolic case. Again the cumulative phase shift $\Delta\varphi$ is represented by the shaded area.

The factor $\alpha = \gamma G_2 \tau_s \xi^2$ in Eq. (2) is quite small and shows the scale of the phase change after a single jump of the spin. $\Delta\varphi$ is a function of random $\{a_j\}$ and therefore itself becomes a random variable. To avoid ambiguity we assume that the phase, $\Delta\varphi$, takes the principal value within an interval of length 2π.

If $P(\varphi) = \int_{-R}^{R} \rho(z_0) p(\varphi|z_0)\,dz_0$ is a probability distribution of random phase shifts, $\Delta\varphi$, then the second moment can be expressed as $\int_{-\infty}^{\infty} \int_{-R}^{R} (\varphi)^2 \rho(z_0) p(\varphi|z_0)\,dz_0\,d\varphi(\{a_i\})$, where $p(\varphi|z_0)$ is a conditional probability function yielding the probability that the phase of a spin which started at z_0 will be found within a phase range, $\Delta\varphi$. In this double integral, interchanging the order of integrating is permissible since the averages are carried out over different independent variables. After averaging over z_0, the variable diffusion coefficient must be treated as a global characteristic of a homogeneous sample in a parabolic magnetic field. On the other hand, if R is the voxel size, then the diffusion coefficient reflects the averaged characteristics of a single voxel.

The ensemble-averaged transverse magnetization is phase modulated and weighted by the coefficient $\langle \exp(i\Delta\varphi(\tau)\rangle$, where $\langle \cdots \rangle$ is an averaging of random variables together with a probability distribution function. Since function $p(\varphi|z_0)$ is not known, then the probability distribution $P(\varphi)$ cannot be found either. This particular problem can be solved by the method of moments [12,15] which states that the unknown density probability function can be recovered fully from its moments if they are limited and the characteristic density probability function is a series

$$\langle \exp(i\Delta\varphi(\tau)) \rangle = \exp\langle i\Delta\varphi(\tau) \rangle. \tag{3}$$

It is clear that the first moment of $\Delta\varphi$ is equal to zero, meaning that the sample magnetization will refocus in the same direction as if no diffusion had taken place. Only the second moment of random value $\Delta\varphi$ stores the dominating information about distribution function. The remaining moments answer the question of how much the distribution deviates from being gaussian [12,13]. According the central limit theorem, the gaussian distribution approximates the unknown density probability with good accuracy when the number of spin jumps is large. Therefore we concentrate our attention on the second moment in the following calculations.

2.1. Translational motion of spins in the presence of a parabolic magnetic field

First consider motion of the spins in a parabolic magnetic field following a 90° RF pulse. In order to evaluate $\langle \Delta\varphi^2(\tau) \rangle$, the phase change in Eq. (2) is squared:

$$\Delta\varphi^2(n\tau_s) = (\gamma G_2 \tau_s \xi^2)^2 [\Sigma_1 + \Sigma_2 + \Sigma_3], \tag{4}$$

where

$$\Sigma_1 = \left(\sum_{m=1}^{n} (n+1-m) a_m^2 \right)^2, \tag{5a}$$

$$\Sigma_2 = \left(\sum_{m=1}^{n} (n+1-m)(A_m - a_m^2) \right)^2, \tag{5b}$$

$$\Sigma_3 = 2\left(\sum_{m=1}^{n} (n+1-m) \sum_{m'=1}^{n} (n+1-m')(A_{m'} - a_{m'}^2) \right)^2. \tag{5c}$$

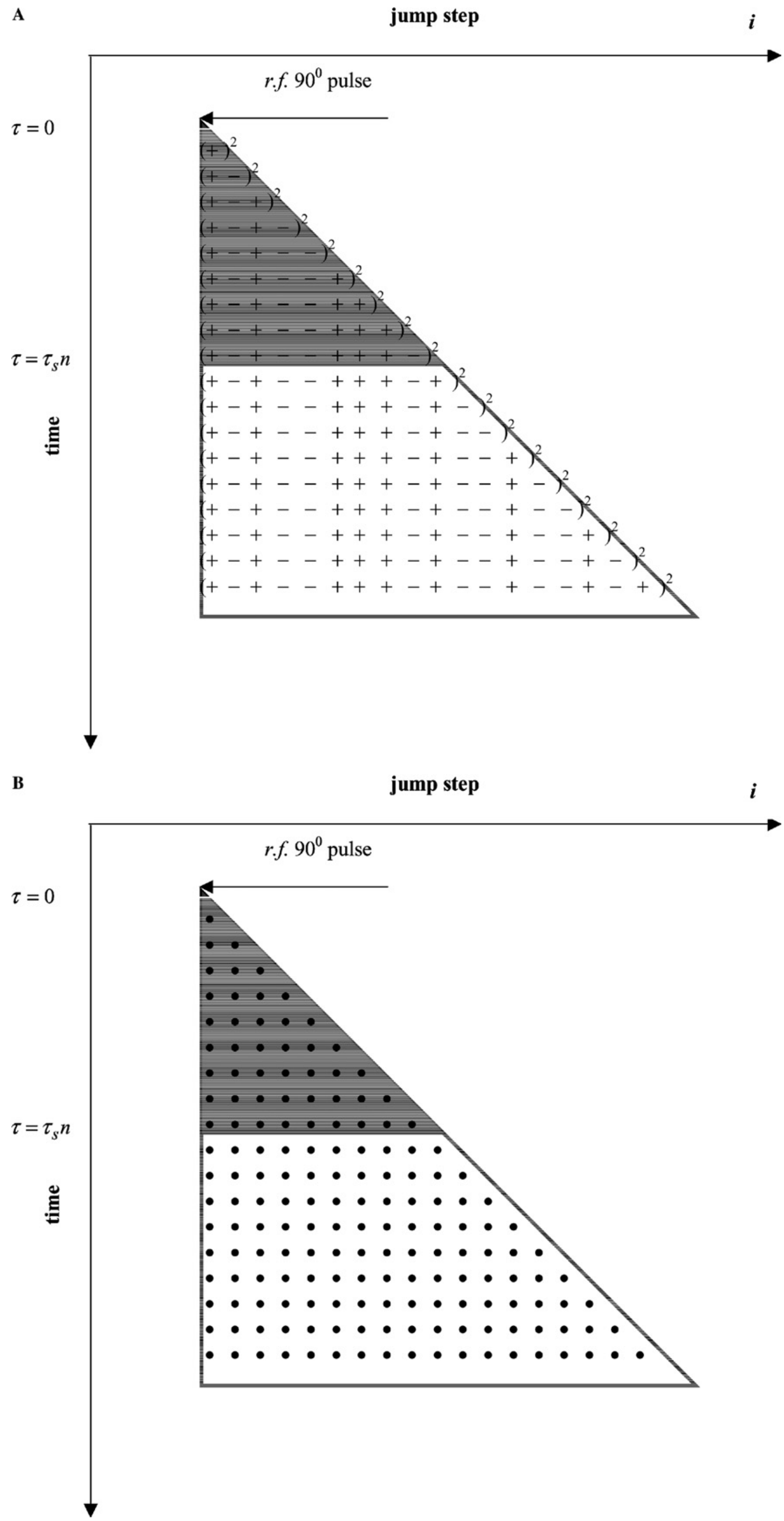

Fig. 2. (A) The sum of jump units along the row followed by squaring is proportional to the precessional frequency change. The sign at the end of each row indicates whether the jump after time interval, τ, is positive or negative with respect to the parabolic magnetic field direction. The cumulative phase shift at time τ in the parabolic magnetic field is given by the sum within the shaded triangle. For the case of a linear magnetic field, a description may be found in [14]. (B) The square of jump units can be represented by the sum of factors $A_i(\cdot)$. The cumulative phase shift is presented in the shaded area.

Thus Eq. (5a) gives $\Sigma_1 = \frac{n^4}{4}(1 + \frac{2}{n} + \frac{1}{n^2})$. When the number of hops, N, is large but $\sqrt{N} \ll \frac{R}{\xi}$ (this assumption is valid for time scales of 1ms in many liquids [7,16]), the latter expression gives

$$\langle \Sigma_1 \rangle_{(n>N)} = \frac{n^4}{4}, \tag{6a}$$

where $\langle \cdots \rangle$ denotes an average over the whole configuration of $\{a_j\}$. A validation of Eq. (6a) can be found in Appendix A. The ensemble average of Eq. (5b) is $\langle \Sigma_2 \rangle = 4\sum_{m=1}^{n}(n-m)^2 m$ and for a large number of hops

$$\langle \Sigma_2 \rangle_{(n>N)} = \frac{n^4}{3}. \tag{6b}$$

Eq. (5c) is exactly zero after the ensemble average procedure. Substituting Eqs. (6a) and (6b) into (4), the squared phase change is:

$$\langle \Delta\varphi^2(n\tau_s) \rangle = \tfrac{7}{12}(\gamma G_2 \tau_s \xi^2 n^2)^2. \tag{7}$$

Defining the self-diffusion coefficient as

$$D = \xi^2 / 2\tau_s. \tag{8}$$

Eq. (7) can be simplified to:

$$\langle \Delta\varphi^2(\tau) \rangle = \tfrac{7}{3}(\gamma G_2 D)^2 \tau^4. \tag{9}$$

Therefore, Eq. (3) can be expressed now as an exponential function of the diffusion factor

$$\langle \exp(\mathrm{i}\Delta\varphi(\tau)) \rangle = \exp\left[-\tfrac{7}{6}(\gamma G_2 D)^2 \tau^4\right]. \tag{10}$$

2.2. Effect on the echo in the presence of a constant parabolic magnetic field

The dephasing effect, which produces an echo at time 2τ in the presence of the constant parabolic magnetic field, must now be considered. The phase diagram for this case is shown in Fig. 3. The 180° RF pulse reverses all phase shifts that existed before time τ. Following the method described above, the cumulative phase change after time 2τ is exactly as in Eq. (2) but with n replaced by $2n$, $c_m = m - 1$, $m = \overline{1, n}$ and

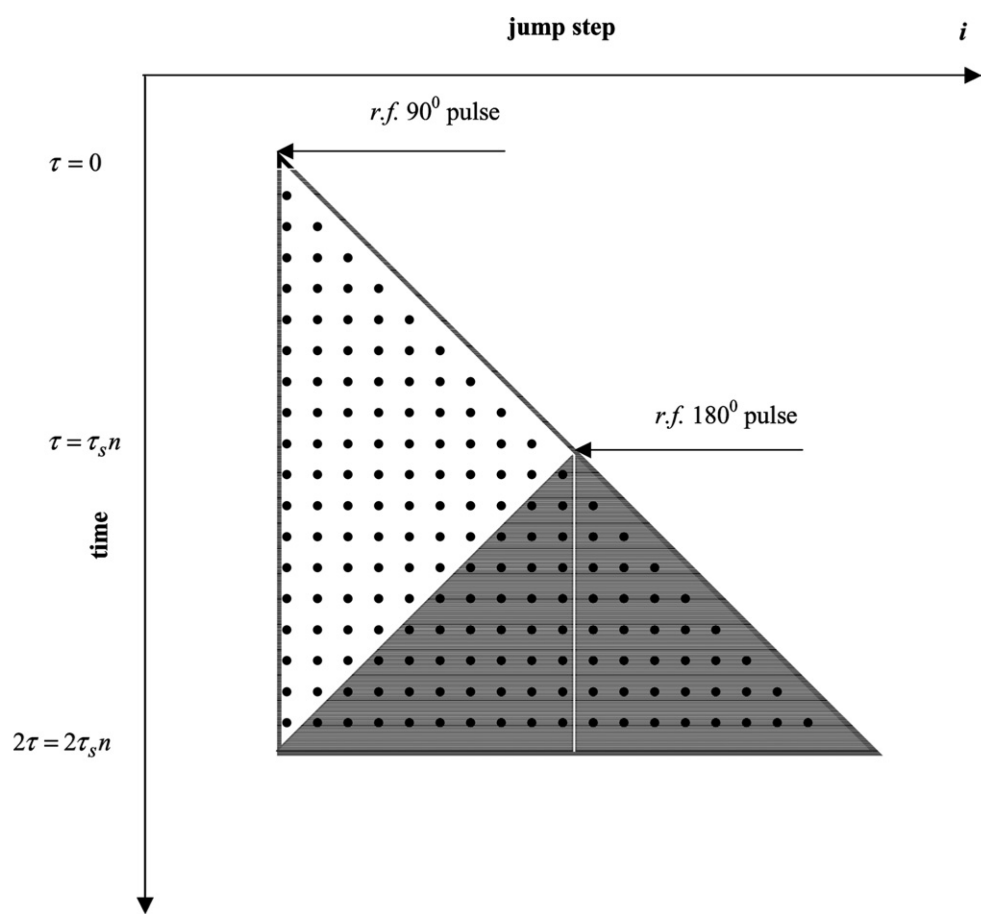

Fig. 3. The cumulative phase shift for spin echo formation at time 2τ in a constant parabolic magnetic field is given by the subtraction of that occurring before and after the 180° RF pulse: trapezium and triangle. After this procedure just two shaded triangles are left (uncorrelated parts). The rearrangement of this double series is treated in [14].

$c_m = 2n - m + 1$, $m = \overline{n+1, 2n}$. In order to evaluate $\langle \Delta\varphi^2(2\tau)\rangle$, the phase change in Eq. (2) can be squared:

$$\Delta\varphi^2(2n\tau_s) = (\gamma G_2 \tau_s \xi^2)^2[\Sigma_1 + \Sigma_2 + \Sigma_3], \tag{11}$$

where

$$\Sigma_1 = \left(\sum_{m=1}^{2n} c_m a_m^2\right)^2, \tag{12a}$$

$$\Sigma_2 = \left(\sum_{m=1}^{2n} c_m (A_m - a_m^2)\right)^2, \tag{12b}$$

$$\Sigma_3 = 2\left(\sum_{m=1}^{2n} c_m \sum_{m'=1}^{2n} c_{m'} (A_{m'} - a_{m'}^2)\right)^2. \tag{12c}$$

Therefore, Eq. (12a) gives

$$\langle \Sigma_1 \rangle = n^4. \tag{13a}$$

Details concerning the averaging of Eq. (12a) are given in Appendix B. The ensemble average of Eq. (12b) is $\langle \Sigma_2 \rangle = 4[\sum_{m=1}^{n}(m-1)^3 + \sum_{m=n+1}^{2n}(2n-m+1)^2(m-1)]$ and for a large number of hops,

$$\langle \Sigma_2 \rangle_{(n>N)} = \frac{8n^4}{3}. \tag{13b}$$

Eq. (12c) equals exactly zero after the ensemble average procedure. Taking Eqs. (13a) and (13b) and substituting into Eq. (11), we obtain:

$$\langle \Delta\varphi^2(2n\tau_s)\rangle = \tfrac{11}{3}(\gamma G_2 \tau_s n^2 \xi^2)^2. \tag{14}$$

Given Eq. (8), Eq. (14) can be written as

$$\langle \Delta\varphi^2(2n\tau_s)\rangle = \tfrac{44}{3}(\gamma G_2 \tau_s^2 n^2 D)^2, \tag{15}$$

and the diffusion factor can be expressed as

$$\langle \exp(\mathrm{i}\Delta\varphi(2\tau))\rangle = \exp\left[-\tfrac{22}{3}(\gamma G_2 D)^2 \tau^4\right]. \tag{16}$$

2.3. Effect on the echo in the presence of pulsed parabolic magnetic field

For a pulsed parabolic magnetic field, it is apparent that the phase shift is obtained by summing two uncorrelated triangular regions and the rectangular one (Fig. 4). If n is the number of hops during the

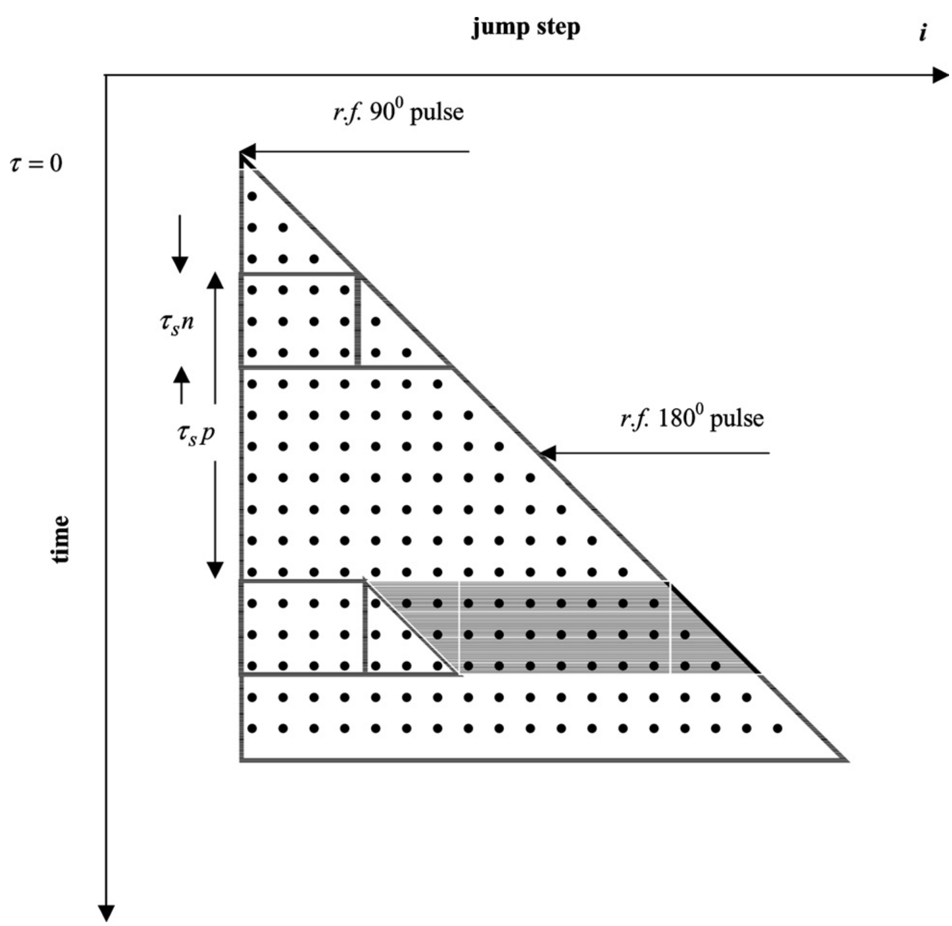

Fig. 4. The cumulative phase shift in a pulsed parabolic magnetic field is given by the sum of factors $A_i(\cdot)$ within the shaded region. The procedure of the phase shift calculation is exactly the same as that discussed in [14].

pulsed parabolic magnetic field pulse and p is a number of hops between two pulses of the parabolic magnetic field, then the following net phase shift calculation is applicable. First, consider the rectangular part:

$$\Delta\varphi_{\mathrm{rect}} = \gamma G_2 \tau_s \xi^2 n \sum_{m=n}^{p} A_m. \tag{17}$$

The square and ensemble-average of the result in Eq. (17) gives

$$\langle \Delta\varphi^2 \rangle_{\mathrm{rect}} = (\gamma G_2 \tau_s \xi^2 n)^2 (\langle \Sigma_1 \rangle + \langle \Sigma_2 \rangle), \tag{18}$$

where

$$\langle \Sigma_1 \rangle = n^2 (p-n)^2, \tag{19a}$$

$$\langle \Sigma_2 \rangle = 2n^2 (p^2 - n^2). \tag{19b}$$

Intermediate steps of this calculation are given in Appendix C. Taking into account Eq. (9), the triangular parts in Fig. 4 give

$$\langle \Delta\varphi^2 \rangle_{\mathrm{tr}} = \tfrac{7}{6} (\gamma G_2 \tau_s \xi^2 n^2)^2. \tag{20}$$

If $\delta = n\tau_s$ and $\Delta = p\tau_s$ (the time delay Δ should not be confused with the notation for phase fluctuation), then the total diffusion factor is

$$\langle \exp(\mathrm{i}\Delta\varphi) \rangle = \exp\left[-4(\gamma G_2 D)^2 \delta^2 \Delta \left(\frac{3}{2}\Delta - \delta + \frac{1}{12\Delta}\delta^2\right)\right], \tag{21}$$

where Eq. (8) has been used. If we suppose that $\Delta \gg \delta^2/12$, then Eq. (21) can be rewritten as

$$\langle \exp(\mathrm{i}\Delta\varphi) \rangle = \exp\left[-6(\gamma G_2 D)^2 \delta^2 \Delta \left(\Delta - \tfrac{2}{3}\delta\right)\right]. \tag{22}$$

When diffusion attenuation occurs mostly during time Δ, then a "frozen" fluctuation scale can be introduced as $2\Delta D = \overline{z^2}$ which leads to a particular form for Eq. (22):

$$\langle \exp(\mathrm{i}\Delta\varphi) \rangle = \exp\left[-3(\gamma G_2)^2 D \overline{z^2} \delta^2 \left(\Delta - \tfrac{2}{3}\delta\right)\right]. \tag{23}$$

3. Comparative analysis of results obtained

Estimation of the diffusion decay coefficients is a crucial factor in many theoretical studies since it provides an insight into questions about the scales of the effect. If the magnitude of the signal is large enough relative to hardware noise effects, then the applicable aspects of the diffusion phenomena can be demonstrated. In Table 1, diffusion factors obtained with different approaches have been summarized. All the other methods were developed for pore size estimation (local magnetic field variation), so our results are comparable only in the case of the so-called "short-time" limit (unrestricted free diffusion). Here the characteristic times are $\tau \ll \frac{R^2}{D}$ and $\delta, \Delta \ll \frac{R^2}{D}$, meaning that the size, R, of the sample (or the size of the pores or the voxel size) does not play a role. Moreover, in the previous studies the micro changes of $(\overrightarrow{\nabla} B)^2$ were taken into consideration and averaged. In the current work, we have taken account of the evolution of phase shift in the presence of an external nonlinear field for homogeneous samples. Nevertheless, analogous results exist and this offers the possibility for comparison. The calculated attenuation factors can be applied in the case of homogeneous voxels in medical applications of diffusion imaging. As can be seen from Table 1, the differences in diffusion attenuation factors in the presence of parabolic magnetic fields calculated with other methods and those resulting from the present work are within 30%. This can be considered as one of the possible explanations of overestimation diffusion coefficients in some NMR experiments and in medical applications of diffusion imaging where such contributions have been wholly neglected until very recently [1].

Eq. (2) is similar to the fluctuations of phase shift in the presence of a linear magnetic field. That is why the effective media approach (variation of local field method) gives results similar to the more complex methods such as the gaussian phase approximation, the Green's function method, and the propagator method. If the coefficients $\{A_i\}$ in Eq. (2) are treated as random

Table 1
Diffusion factors $|\ln(M/M_0)|$ for different models in the case of $\tau < T_2^*, T_2, T_1$

	$\frac{\pi}{2}$ RF pulse in a steady parabolic magnetic field	$\frac{\pi}{2} - \tau - \pi - \tau$ RF pulses in a steady parabolic magnetic field	$\frac{\pi}{2} - \tau - \pi - \tau$ RF pulses in a pulsed parabolic magnetic field
Variation of local field method	—	$\frac{8}{3}\gamma^2 G_2^2 D R^2 \tau^3$	—
Green's function method, gaussian phase approximation	$\frac{2}{3}\gamma^2 G_2^2 D^2 \tau^4$	$\frac{1}{12}\gamma^2 D (2\tau)^3 \overline{(\nabla B)^2}$	$\gamma^2 D \delta^2 \left(\Delta - \frac{1}{3}\delta\right) \overline{(\nabla B)^2}$
Propagator method	—	$\frac{2}{3}\gamma^2 D \tau^3 \overline{(\nabla B)^2}$	$4\gamma^2 G_2^2 D \delta^2 \overline{z^2} \left(\Delta - \frac{1}{3}\delta\right)$
Present work	$\frac{7}{6}\gamma^2 G_2^2 D^2 \tau^4$	$\frac{22}{3}\gamma^2 G_2^2 D^2 \tau^4$	$6\gamma^2 G_2^2 D^2 \delta^2 \Delta \left(\Delta - \frac{2}{3}\delta\right)$
			$3(\gamma G_2)^2 D \overline{z^2} \delta^2 \left(\Delta - \frac{2}{3}\delta\right)$

T_2^*, T_2, and T_1 are the transverse and longitudinal relaxation times.

variables, generated by linear transformation of variables $\{a_i\}$, then it is possible to consider the problem as in a linear magnetic field but with the renormalized coefficient of magnetic field $G_1^* \to G_1 + G_2 z^2$. Representation of phase fluctuations through independent collective variables, however, leads to the singularity problem after using geometric progression (see [7]). In principle, $\{A_i\} = f(\{a_i\})$ is a non-linear transformation by definition and it is difficult to find $\{a_i\} = f^{-1}(\{A_i\})$ analytically. Of course, in the limit the effective media (classical) results can be obtained if we neglect Σ_1 in Eqs. (4), (11), and (18).

4. Discussion

We have presented the solution of the free diffusion of spins in the presence of parabolic magnetic fields. The treatment of such a problem gives heavier diffusion attenuation factors than those obtained from the Green's function method, gaussian phase approximation, and variation of local field method. We assume these discrepancies originate from the fact that we have not neglected fourth order terms such as $\langle a_i^2 a_j^2 \rangle_{i \neq j}$. In addition, our calculation show that non-linear magnetic fields increase attenuation due to diffusion effects; this can be clearly seen from a comparison of Eq. (21) and the well-known Stejskal–Tanner equation [17].

Appendix A.

Let us consider the averaging procedure for Σ_1 (Eq. (5a)). Because $a_i^2 = 1$, we have to find a sum $(\sum_{m=1}^{n}(n+1-m))^2$. This is nothing else other than a square of an arithmetic series. According to the well-known formula $\sum_{m=1}^{n}(n+1-m) = n + \frac{1}{2}(n-1)n$, the square of it gives

$$\left(n + \frac{1}{2}(n-1)n\right)^2 = \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4} \underset{n>N}{\to} \frac{n^4}{4}, \tag{A.1}$$

which corresponds to Eq. (6a). The averaging procedure for Eqs. (5b) and (5c) is more complex and demands account of the terms $a_i^2 a_j^2$ and $a_i^2 a_j a_k$ where $i \neq j \neq k$. After the averaging procedure, only terms of the first kind $\langle a_i^2 a_j^2 \rangle \neq 0$ give a non-zero contribution in the second moment of the phase shift.

Appendix B.

As mentioned in Appendix A we can apply arithmetic series summation rules to calculate the average value of Eq. (12a):

$$\begin{aligned}\Sigma_1 &= \left(\sum_{m=1}^{2n} c_m\right)^2 \\ &= \left(\sum_{m=1}^{n}(m-1) + \sum_{m=n+1}^{2n}(2n-m+1)\right)^2 \\ &= \left(n\left(\frac{n-1}{2}\right) + n\left(\frac{n+1}{2}\right)\right) \underset{n>N}{\to} n^4.\end{aligned} \tag{B.1}$$

The averaging procedure for Eqs. (12b) and (12c) is the same and involves calculation of $\langle a_i^2 a_j^2 \rangle \neq 0$, $\langle a_i^2 a_j a_k \rangle = 0$, and $\langle a_i a_j a_k a_l \rangle = 0$, where $i \neq j \neq k \neq l$.

Appendix C.

For the rectangular part in Fig. 4 we consider the square of the sum $(\sum_{m=n}^{p} A_m)^2 = (\sum_{m=n}^{p}(2a_m \sum_{k=0}^{i-1} a_k + 1))^2$. Opening of the brackets leads to different kind of forth order combinations of random variable a_i. Only even order combinations give non-zero terms in averaging procedure.

References

[1] R. Bammer, M. Markl, A. Barnett, B. Acar, M.T. Alley, N.J. Pelc, G.H. Glover, M.E. Moseley, Analysis and generalized correction of the effect of spatial gradient field distortions in diffusion-weighted imaging, MRM 50 (2003) 560–569.
[2] D. LeBihan (Ed.), Magnetic Resonance Imaging of Diffusion and Perfusion: Applications to Functional Imaging, Lippincott–Raven Press, New York, 1995.
[3] H.Y. Carr, E.M. Purcell, Effects of diffusion on free precession in nuclear magnetic resonance experiments, Phys. Rev. 94 (3) (1954) 630–638.
[4] P. Bendel, Spin-echo attenuation by diffusion in nonuniform field gradients, J. Magn. Reson. 86 (1990) 509–515.
[5] L.J. Zielinski, P.N. Sen, Relaxation of nuclear magnetization in a nonuniform magnetic field gradient and in a restricted geometry, J. Magn. Reson. 147 (2000) 95–103.
[6] M.D. Huerlimann, Effective gradients in porous media due to susceptibility differences, J. Magn. Reson. 131 (1998) 232–240.
[7] J.C. Tarczon, W.P. Halperin, Interpretation of NMR diffusion measurements in uniform- and nonuniform-field profiles, Phys. Rev. B 32 (5) (1985) 2798–2806.
[8] P. Doussal, P.N. Sen, Decay of nuclear magnetization by diffusion in a parabolic magnetic field: an exactly solvable model, Phys. Rev. B 46 (6) (1992) 3465–3486.
[9] G. Lin, Z. Chen, A novel propagator approach for NMR signal attenuation due to anisotropic diffusion under various magnetic field gradients, Chem. Phys. Lett. 335 (2001) 249–256.
[10] D.J. Bergman, K.J. Dunn, NMR of diffusing atoms in a periodic porous medium in the presence of a nonuniform magnetic field, Phys. Rev. E 52 (6) (1995) 6516–6534.
[11] J.G. Kemeny, J.L. Snell, Finite Markov Chains, Princeton, New Jersey, 1960.
[12] N.I. Ahiezer, M. Krein, Some questions in the theory of moments, Am. Math. Soc. (1962).

[13] J.V. Uspensky, Introduction to Mathematical Probability, McGraw-Hill, New York, 1937.
[14] P.T. Callaghan, Principles of Nuclear Magnetic Resonance Microscopy, Clarendon Press, Oxford, 1991.
[15] B.V. Gnedenko, Course of Probability Theory, Fizmatgiz, Moscow, 1963.
[16] I.Z. Fisher, A. Rice, Statistical Theory of Liquids, University of Chicago Press, Chicago, 1964.
[17] E.O. Stejskal, J.E. Tanner, Spin diffusion measurements: spin echoes in the presence of a time-dependent field gradient, J. Chem. Phys. 42 (1965) 288–302.